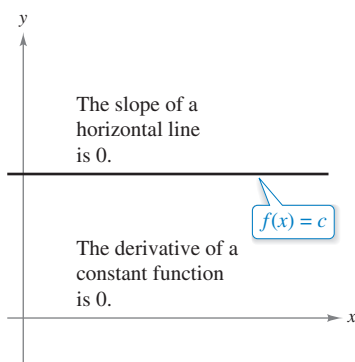


2.2 Basic Differentiation Rules and Rates of Change

- Find the derivative of a function using the Constant Rule.
- Find the derivative of a function using the Power Rule.
- Find the derivative of a function using the Constant Multiple Rule.
- Find the derivative of a function using the Sum and Difference Rules.
- Find the derivatives of the sine function and of the cosine function.
- Use derivatives to find rates of change.

The Constant Rule

In Section 2.1, you used the limit definition to find derivatives. In this and the next two sections, you will be introduced to several “differentiation rules” that allow you to find derivatives without the *direct* use of the limit definition.



Notice that the Constant Rule is equivalent to saying that the slope of a horizontal line is 0. This demonstrates the relationship between slope and derivative.

Figure 2.14

THEOREM 2.2 The Constant Rule

The derivative of a constant function is 0. That is, if c is a real number, then

$$\frac{d}{dx}[c] = 0. \quad (\text{See Figure 2.14.})$$

Proof Let $f(x) = c$. Then, by the limit definition of the derivative,

$$\begin{aligned} \frac{d}{dx}[c] &= f'(x) \\ &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{c - c}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} 0 \\ &= 0. \end{aligned}$$

See LarsonCalculus.com for Bruce Edwards's video of this proof.

EXAMPLE 1

Using the Constant Rule

Function	Derivative
a. $y = 7$	$dy/dx = 0$
b. $f(x) = 0$	$f'(x) = 0$
c. $s(t) = -3$	$s'(t) = 0$
d. $y = k\pi^2$, k is constant	$y' = 0$

Exploration

Writing a Conjecture Use the definition of the derivative given in Section 2.1 to find the derivative of each function. What patterns do you see? Use your results to write a conjecture about the derivative of $f(x) = x^n$.

- | | | |
|-----------------|---------------------|--------------------|
| a. $f(x) = x^1$ | b. $f(x) = x^2$ | c. $f(x) = x^3$ |
| d. $f(x) = x^4$ | e. $f(x) = x^{1/2}$ | f. $f(x) = x^{-1}$ |

The Power Rule

Before proving the next rule, it is important to review the procedure for expanding a binomial.

$$(x + \Delta x)^2 = x^2 + 2x\Delta x + (\Delta x)^2$$

$$(x + \Delta x)^3 = x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3$$

$$(x + \Delta x)^4 = x^4 + 4x^3\Delta x + 6x^2(\Delta x)^2 + 4x(\Delta x)^3 + (\Delta x)^4$$

$$(x + \Delta x)^5 = x^5 + 5x^4\Delta x + 10x^3(\Delta x)^2 + 10x^2(\Delta x)^3 + 5x(\Delta x)^4 + (\Delta x)^5$$

The general binomial expansion for a positive integer n is

$$(x + \Delta x)^n = x^n + nx^{n-1}(\Delta x) + \underbrace{\frac{n(n-1)x^{n-2}}{2}(\Delta x)^2 + \cdots + (\Delta x)^n}_{(\Delta x)^2 \text{ is a factor of these terms.}}$$

This binomial expansion is used in proving a special case of the Power Rule.

THEOREM 2.3 The Power Rule

If n is a rational number, then the function $f(x) = x^n$ is differentiable and

$$\frac{d}{dx}[x^n] = nx^{n-1}.$$

For f to be differentiable at $x = 0$, n must be a number such that x^{n-1} is defined on an interval containing 0.

•• **REMARK** From Example 7 in Section 2.1, you know that the function $f(x) = x^{1/3}$ is defined at $x = 0$, but is not differentiable at $x = 0$. This is because $x^{-2/3}$ is not defined on an interval containing 0.

Proof If n is a positive integer greater than 1, then the binomial expansion produces

$$\begin{aligned} \frac{d}{dx}[x^n] &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^n - x^n}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x^n + nx^{n-1}(\Delta x) + \frac{n(n-1)x^{n-2}}{2}(\Delta x)^2 + \cdots + (\Delta x)^n - x^n}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \left[nx^{n-1} + \frac{n(n-1)x^{n-2}}{2}(\Delta x) + \cdots + (\Delta x)^{n-1} \right] \\ &= nx^{n-1} + 0 + \cdots + 0 \\ &= nx^{n-1}. \end{aligned}$$

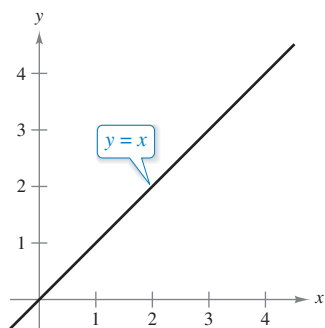
This proves the case for which n is a positive integer greater than 1. It is left to you to prove the case for $n = 1$. Example 7 in Section 2.3 proves the case for which n is a negative integer. In Exercise 71 in Section 2.5, you are asked to prove the case for which n is rational. (In Section 5.5, the Power Rule will be extended to cover irrational values of n .)

See LarsonCalculus.com for Bruce Edwards's video of this proof.

When using the Power Rule, the case for which $n = 1$ is best thought of as a separate differentiation rule. That is,

$$\frac{d}{dx}[x] = 1.$$

Power Rule when $n = 1$



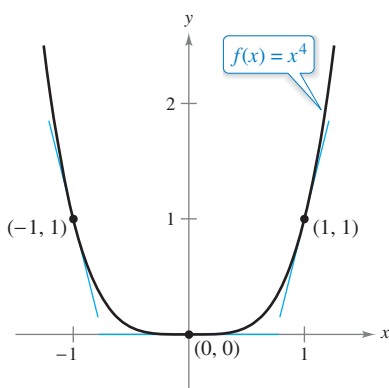
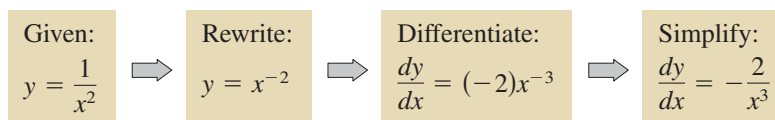
The slope of the line $y = x$ is 1.
Figure 2.15

This rule is consistent with the fact that the slope of the line $y = x$ is 1, as shown in Figure 2.15.

EXAMPLE 2**Using the Power Rule**

Function	Derivative
a. $f(x) = x^3$	$f'(x) = 3x^2$
b. $g(x) = \sqrt[3]{x}$	$g'(x) = \frac{d}{dx}[x^{1/3}] = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$
c. $y = \frac{1}{x^2}$	$\frac{dy}{dx} = \frac{d}{dx}[x^{-2}] = (-2)x^{-3} = -\frac{2}{x^3}$

In Example 2(c), note that *before* differentiating, $1/x^2$ was rewritten as x^{-2} . Rewriting is the first step in *many* differentiation problems.



Note that the slope of the graph is negative at the point $(-1, 1)$, the slope is zero at the point $(0, 0)$, and the slope is positive at the point $(1, 1)$.

Figure 2.16**EXAMPLE 3****Finding the Slope of a Graph**

•••▶ See LarsonCalculus.com for an interactive version of this type of example.

Find the slope of the graph of

$$f(x) = x^4$$

for each value of x .

- a. $x = -1$ b. $x = 0$ c. $x = 1$

Solution The slope of a graph at a point is the value of the derivative at that point. The derivative of f is $f'(x) = 4x^3$.

- a. When $x = -1$, the slope is $f'(-1) = 4(-1)^3 = -4$. Slope is negative.
 b. When $x = 0$, the slope is $f'(0) = 4(0)^3 = 0$. Slope is zero.
 c. When $x = 1$, the slope is $f'(1) = 4(1)^3 = 4$. Slope is positive.

See Figure 2.16.

EXAMPLE 4**Finding an Equation of a Tangent Line**

•••▶ See LarsonCalculus.com for an interactive version of this type of example.

Find an equation of the tangent line to the graph of $f(x) = x^2$ when $x = -2$.

Solution To find the *point* on the graph of f , evaluate the original function at $x = -2$.

$$(-2, f(-2)) = (-2, 4) \quad \text{Point on graph}$$

To find the *slope* of the graph when $x = -2$, evaluate the derivative, $f'(x) = 2x$, at $x = -2$.

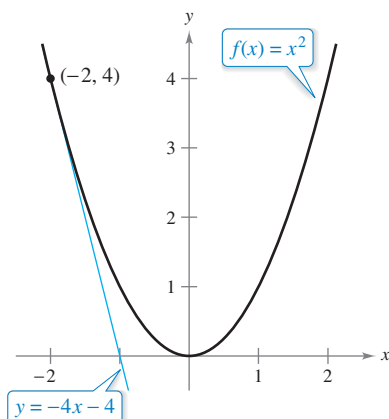
$$m = f'(-2) = -4 \quad \text{Slope of graph at } (-2, 4)$$

Now, using the point-slope form of the equation of a line, you can write

$$y - y_1 = m(x - x_1) \quad \text{Point-slope form}$$

$$y - 4 = -4[x - (-2)] \quad \text{Substitute for } y_1, m, \text{ and } x_1.$$

$$y = -4x - 4. \quad \text{Simplify.}$$



The line $y = -4x - 4$ is tangent to the graph of $f(x) = x^2$ at the point $(-2, 4)$.

Figure 2.17

See Figure 2.17.

The Constant Multiple Rule

THEOREM 2.4 The Constant Multiple Rule

If f is a differentiable function and c is a real number, then cf is also differentiable and $\frac{d}{dx}[cf(x)] = cf'(x)$.

Proof

$$\begin{aligned}\frac{d}{dx}[cf(x)] &= \lim_{\Delta x \rightarrow 0} \frac{cf(x + \Delta x) - cf(x)}{\Delta x} && \text{Definition of derivative} \\ &= \lim_{\Delta x \rightarrow 0} c \left[\frac{f(x + \Delta x) - f(x)}{\Delta x} \right] \\ &= c \left[\lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \right] && \text{Apply Theorem 1.2.} \\ &= cf'(x)\end{aligned}$$

See LarsonCalculus.com for Bruce Edwards's video of this proof.

Informally, the Constant Multiple Rule states that constants can be factored out of the differentiation process, even when the constants appear in the denominator.

$$\begin{aligned}\frac{d}{dx}[cf(x)] &= c \frac{d}{dx}[f(x)] = cf'(x) \\ \frac{d}{dx}\left[\frac{f(x)}{c}\right] &= \frac{d}{dx}\left[\left(\frac{1}{c}\right)f(x)\right] = \left(\frac{1}{c}\right) \frac{d}{dx}[f(x)] = \left(\frac{1}{c}\right)f'(x)\end{aligned}$$

EXAMPLE 5 Using the Constant Multiple Rule

Function

Derivative

- | | |
|-----------------------------------|---|
| a. $y = 5x^3$ | $\frac{dy}{dx} = \frac{d}{dx}[5x^3] = 5 \frac{d}{dx}[x^3] = 5(3)x^2 = 15x^2$ |
| b. $y = \frac{2}{x}$ | $\frac{dy}{dx} = \frac{d}{dx}[2x^{-1}] = 2 \frac{d}{dx}[x^{-1}] = 2(-1)x^{-2} = -\frac{2}{x^2}$ |
| c. $f(t) = \frac{4t^2}{5}$ | $f'(t) = \frac{d}{dt}\left[\frac{4}{5}t^2\right] = \frac{4}{5} \frac{d}{dt}[t^2] = \frac{4}{5}(2t) = \frac{8}{5}t$ |
| d. $y = 2\sqrt{x}$ | $\frac{dy}{dx} = \frac{d}{dx}[2x^{1/2}] = 2\left(\frac{1}{2}x^{-1/2}\right) = x^{-1/2} = \frac{1}{\sqrt{x}}$ |
| e. $y = \frac{1}{2\sqrt[3]{x^2}}$ | $\frac{dy}{dx} = \frac{d}{dx}\left[\frac{1}{2}x^{-2/3}\right] = \frac{1}{2}\left(-\frac{2}{3}\right)x^{-5/3} = -\frac{1}{3x^{5/3}}$ |
| f. $y = -\frac{3x}{2}$ | $y' = \frac{d}{dx}\left[-\frac{3}{2}x\right] = -\frac{3}{2}(1) = -\frac{3}{2}$ |

.....►
REMARK Before differentiating functions involving radicals, rewrite the function with rational exponents.

The Constant Multiple Rule and the Power Rule can be combined into one rule. The combination rule is

$$\frac{d}{dx}[cx^n] = cnx^{n-1}.$$

EXAMPLE 6 Using Parentheses When Differentiating

Original Function	Rewrite	Differentiate	Simplify
a. $y = \frac{5}{2x^3}$	$y = \frac{5}{2}(x^{-3})$	$y' = \frac{5}{2}(-3x^{-4})$	$y' = -\frac{15}{2x^4}$
b. $y = \frac{5}{(2x)^3}$	$y = \frac{5}{8}(x^{-3})$	$y' = \frac{5}{8}(-3x^{-4})$	$y' = -\frac{15}{8x^4}$
c. $y = \frac{7}{3x^{-2}}$	$y = \frac{7}{3}(x^2)$	$y' = \frac{7}{3}(2x)$	$y' = \frac{14x}{3}$
d. $y = \frac{7}{(3x)^{-2}}$	$y = 63(x^2)$	$y' = 63(2x)$	$y' = 126x$

The Sum and Difference Rules**THEOREM 2.5 The Sum and Difference Rules**

The sum (or difference) of two differentiable functions f and g is itself differentiable. Moreover, the derivative of $f + g$ (or $f - g$) is the sum (or difference) of the derivatives of f and g .

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x) \quad \text{Sum Rule}$$

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x) \quad \text{Difference Rule}$$

Proof A proof of the Sum Rule follows from Theorem 1.2. (The Difference Rule can be proved in a similar way.)

$$\begin{aligned}
 \frac{d}{dx}[f(x) + g(x)] &= \lim_{\Delta x \rightarrow 0} \frac{[f(x + \Delta x) + g(x + \Delta x)] - [f(x) + g(x)]}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) + g(x + \Delta x) - f(x) - g(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \left[\frac{f(x + \Delta x) - f(x)}{\Delta x} + \frac{g(x + \Delta x) - g(x)}{\Delta x} \right] \\
 &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} + \lim_{\Delta x \rightarrow 0} \frac{g(x + \Delta x) - g(x)}{\Delta x} \\
 &= f'(x) + g'(x)
 \end{aligned}$$

See *LarsonCalculus.com* for Bruce Edwards's video of this proof.

The Sum and Difference Rules can be extended to any finite number of functions. For instance, if $F(x) = f(x) + g(x) - h(x)$, then $F'(x) = f'(x) + g'(x) - h'(x)$.

REMARK In Example 7(c), note that before differentiating,

$$\frac{3x^2 - x + 1}{x}$$

was rewritten as

$$3x - 1 + \frac{1}{x}.$$

EXAMPLE 7 Using the Sum and Difference Rules

Function	Derivative
a. $f(x) = x^3 - 4x + 5$	$f'(x) = 3x^2 - 4$
b. $g(x) = -\frac{x^4}{2} + 3x^3 - 2x$	$g'(x) = -2x^3 + 9x^2 - 2$
c. $y = \frac{3x^2 - x + 1}{x} = 3x - 1 + \frac{1}{x}$	$y' = 3 - \frac{1}{x^2} = \frac{3x^2 - 1}{x^2}$

FOR FURTHER INFORMATION

For the outline of a geometric proof of the derivatives of the sine and cosine functions, see the article “The Spider’s Spacewalk Derivation of $\sin'$ and $\cos'$ ” by Tim Hesterberg in *The College Mathematics Journal*. To view this article, go to *MathArticles.com*.

Derivatives of the Sine and Cosine Functions

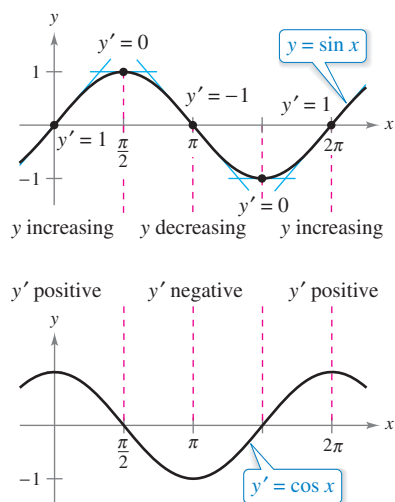
In Section 1.3, you studied the limits

$$\lim_{\Delta x \rightarrow 0} \frac{\sin \Delta x}{\Delta x} = 1 \quad \text{and} \quad \lim_{\Delta x \rightarrow 0} \frac{1 - \cos \Delta x}{\Delta x} = 0.$$

These two limits can be used to prove differentiation rules for the sine and cosine functions. (The derivatives of the other four trigonometric functions are discussed in Section 2.3.)

THEOREM 2.6 Derivatives of Sine and Cosine Functions

$$\frac{d}{dx}[\sin x] = \cos x \qquad \frac{d}{dx}[\cos x] = -\sin x$$



The derivative of the sine function is the cosine function.

Figure 2.18

Proof Here is a proof of the first rule. (The proof of the second rule is left as an exercise [see Exercise 118].)

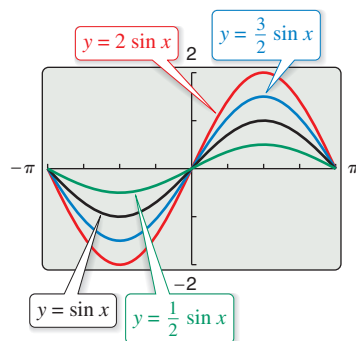
$$\begin{aligned} \frac{d}{dx}[\sin x] &= \lim_{\Delta x \rightarrow 0} \frac{\sin(x + \Delta x) - \sin x}{\Delta x} && \text{Definition of derivative} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\sin x \cos \Delta x + \cos x \sin \Delta x - \sin x}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\cos x \sin \Delta x - (\sin x)(1 - \cos \Delta x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \left[(\cos x) \left(\frac{\sin \Delta x}{\Delta x} \right) - (\sin x) \left(\frac{1 - \cos \Delta x}{\Delta x} \right) \right] \\ &= \cos x \left(\lim_{\Delta x \rightarrow 0} \frac{\sin \Delta x}{\Delta x} \right) - \sin x \left(\lim_{\Delta x \rightarrow 0} \frac{1 - \cos \Delta x}{\Delta x} \right) \\ &= (\cos x)(1) - (\sin x)(0) \\ &= \cos x \end{aligned}$$

This differentiation rule is shown graphically in Figure 2.18. Note that for each x , the slope of the sine curve is equal to the value of the cosine.

See *LarsonCalculus.com* for Bruce Edwards’s video of this proof.

EXAMPLE 8**Derivatives Involving Sines and Cosines**

•••► See *LarsonCalculus.com* for an interactive version of this type of example.



$$\frac{d}{dx}[a \sin x] = a \cos x$$

Figure 2.19

Function	Derivative
a. $y = 2 \sin x$	$y' = 2 \cos x$
b. $y = \frac{\sin x}{2} = \frac{1}{2} \sin x$	$y' = \frac{1}{2} \cos x = \frac{\cos x}{2}$
c. $y = x + \cos x$	$y' = 1 - \sin x$
d. $\cos x - \frac{\pi}{3} \sin x$	$-\sin x - \frac{\pi}{3} \cos x$

► **TECHNOLOGY** A graphing utility can provide insight into the interpretation of a derivative. For instance, Figure 2.19 shows the graphs of

- $y = a \sin x$
- for $a = \frac{1}{2}, 1, \frac{3}{2},$ and 2 . Estimate the slope of each graph at the point $(0, 0)$. Then verify your estimates analytically by evaluating the derivative of each function when $x = 0$.

Rates of Change

You have seen how the derivative is used to determine slope. The derivative can also be used to determine the rate of change of one variable with respect to another. Applications involving rates of change, sometimes referred to as instantaneous rates of change, occur in a wide variety of fields. A few examples are population growth rates, production rates, water flow rates, velocity, and acceleration.

A common use for rate of change is to describe the motion of an object moving in a straight line. In such problems, it is customary to use either a horizontal or a vertical line with a designated origin to represent the line of motion. On such lines, movement to the right (or upward) is considered to be in the positive direction, and movement to the left (or downward) is considered to be in the negative direction.

The function s that gives the position (relative to the origin) of an object as a function of time t is called a **position function**. If, over a period of time Δt , the object changes its position by the amount

$$\Delta s = s(t + \Delta t) - s(t)$$

then, by the familiar formula

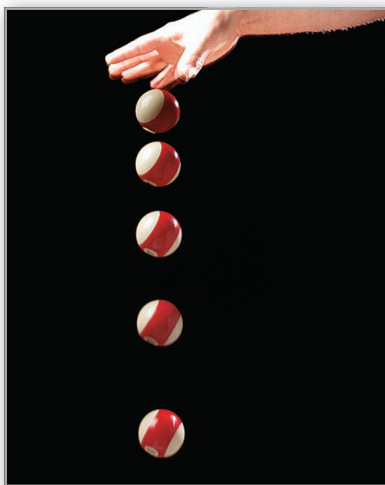
$$\text{Rate} = \frac{\text{distance}}{\text{time}}$$

the **average velocity** is

$$\frac{\text{Change in distance}}{\text{Change in time}} = \frac{\Delta s}{\Delta t}$$

Average velocity

EXAMPLE 9 Finding Average Velocity of a Falling Object



Time-lapse photograph of a free-falling billiard ball

A billiard ball is dropped from a height of 100 feet. The ball's height s at time t is the position function

$$s = -16t^2 + 100 \quad \text{Position function}$$

where s is measured in feet and t is measured in seconds. Find the average velocity over each of the following time intervals.

- a. $[1, 2]$ b. $[1, 1.5]$ c. $[1, 1.1]$

Solution

- a. For the interval $[1, 2]$, the object falls from a height of $s(1) = -16(1)^2 + 100 = 84$ feet to a height of $s(2) = -16(2)^2 + 100 = 36$ feet. The average velocity is

$$\frac{\Delta s}{\Delta t} = \frac{36 - 84}{2 - 1} = \frac{-48}{1} = -48 \text{ feet per second.}$$

- b. For the interval $[1, 1.5]$, the object falls from a height of 84 feet to a height of $s(1.5) = -16(1.5)^2 + 100 = 64$ feet. The average velocity is

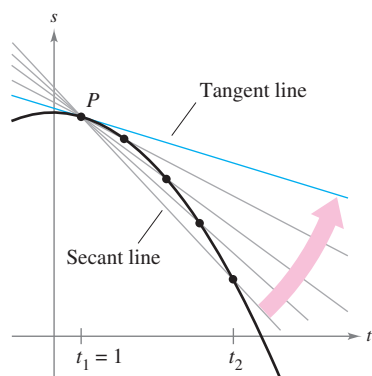
$$\frac{\Delta s}{\Delta t} = \frac{64 - 84}{1.5 - 1} = \frac{-20}{0.5} = -40 \text{ feet per second.}$$

- c. For the interval $[1, 1.1]$, the object falls from a height of 84 feet to a height of $s(1.1) = -16(1.1)^2 + 100 = 80.64$ feet. The average velocity is

$$\frac{\Delta s}{\Delta t} = \frac{80.64 - 84}{1.1 - 1} = \frac{-3.36}{0.1} = -33.6 \text{ feet per second.}$$

Note that the average velocities are *negative*, indicating that the object is moving downward.

Richard Megna/Fundamental Photographs



The average velocity between t_1 and t_2 is the slope of the secant line, and the instantaneous velocity at t_1 is the slope of the tangent line.

Figure 2.20

Suppose that in Example 9, you wanted to find the *instantaneous* velocity (or simply the velocity) of the object when $t = 1$. Just as you can approximate the slope of the tangent line by calculating the slope of the secant line, you can approximate the velocity at $t = 1$ by calculating the average velocity over a small interval $[1, 1 + \Delta t]$ (see Figure 2.20). By taking the limit as Δt approaches zero, you obtain the velocity when $t = 1$. Try doing this—you will find that the velocity when $t = 1$ is -32 feet per second.

In general, if $s = s(t)$ is the position function for an object moving along a straight line, then the **velocity** of the object at time t is

$$v(t) = \lim_{\Delta t \rightarrow 0} \frac{s(t + \Delta t) - s(t)}{\Delta t} = s'(t).$$

Velocity function

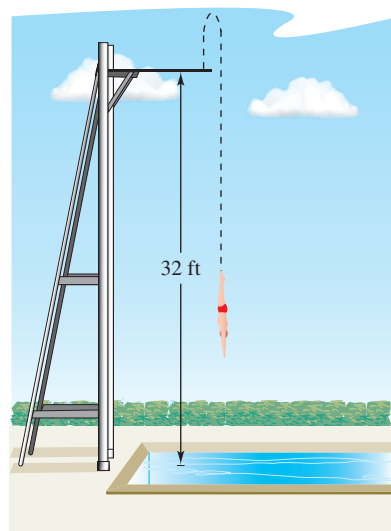
In other words, the velocity function is the derivative of the position function. Velocity can be negative, zero, or positive. The **speed** of an object is the absolute value of its velocity. Speed cannot be negative.

The position of a free-falling object (neglecting air resistance) under the influence of gravity can be represented by the equation

$$s(t) = \frac{1}{2}gt^2 + v_0t + s_0$$

Position function

where s_0 is the initial height of the object, v_0 is the initial velocity of the object, and g is the acceleration due to gravity. On Earth, the value of g is approximately -32 feet per second per second or -9.8 meters per second per second.



Velocity is positive when an object is rising, and is negative when an object is falling. Notice that the diver moves upward for the first half-second because the velocity is positive for $0 < t < \frac{1}{2}$. When the velocity is 0, the diver has reached the maximum height of the dive.

Figure 2.21

EXAMPLE 10 Using the Derivative to Find Velocity

At time $t = 0$, a diver jumps from a platform diving board that is 32 feet above the water (see Figure 2.21). Because the initial velocity of the diver is 16 feet per second, the position of the diver is

$$s(t) = -16t^2 + 16t + 32$$

Position function

where s is measured in feet and t is measured in seconds.

- When does the diver hit the water?
- What is the diver's velocity at impact?

Solution

- To find the time t when the diver hits the water, let $s = 0$ and solve for t .

$$-16t^2 + 16t + 32 = 0$$

Set position function equal to 0.

$$-16(t + 1)(t - 2) = 0$$

Factor.

$$t = -1 \text{ or } 2$$

Solve for t .

Because $t \geq 0$, choose the positive value to conclude that the diver hits the water at $t = 2$ seconds.

- The velocity at time t is given by the derivative

$$s'(t) = -32t + 16.$$

Velocity function

So, the velocity at time $t = 2$ is

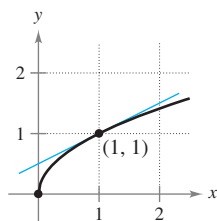
$$s'(2) = -32(2) + 16 = -48 \text{ feet per second.}$$

2.2 Exercises

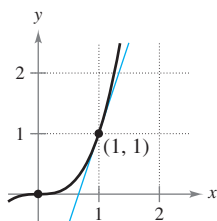
See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Estimating Slope In Exercises 1 and 2, use the graph to estimate the slope of the tangent line to $y = x^n$ at the point (1, 1). Verify your answer analytically. To print an enlarged copy of the graph, go to MathGraphs.com.

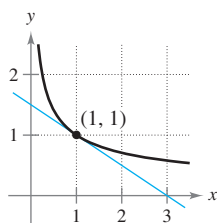
1. (a) $y = x^{1/2}$



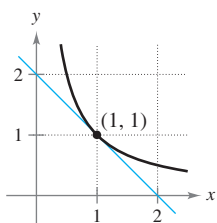
(b) $y = x^3$



2. (a) $y = x^{-1/2}$



(b) $y = x^{-1}$



Finding a Derivative In Exercises 3–24, use the rules of differentiation to find the derivative of the function.

3. $y = 12$

4. $f(x) = -9$

5. $y = x^7$

6. $y = x^{12}$

7. $y = \frac{1}{x^5}$

8. $y = \frac{3}{x^7}$

9. $f(x) = \sqrt[5]{x}$

10. $g(x) = \sqrt[4]{x}$

11. $f(x) = x + 11$

12. $g(x) = 6x + 3$

13. $f(t) = -2t^2 + 3t - 6$

14. $y = t^2 - 3t + 1$

15. $g(x) = x^2 + 4x^3$

16. $y = 4x - 3x^3$

17. $s(t) = t^3 + 5t^2 - 3t + 8$

18. $y = 2x^3 + 6x^2 - 1$

19. $y = \frac{\pi}{2} \sin \theta - \cos \theta$

20. $g(t) = \pi \cos t$

21. $y = x^2 - \frac{1}{2} \cos x$

22. $y = 7 + \sin x$

23. $y = \frac{1}{x} - 3 \sin x$

24. $y = \frac{5}{(2x)^3} + 2 \cos x$

Rewriting a Function Before Differentiating In Exercises 25–30, complete the table to find the derivative of the function.

Original Function	Rewrite	Differentiate	Simplify
25. $y = \frac{5}{2x^2}$			
26. $y = \frac{3}{2x^4}$			
27. $y = \frac{6}{(5x)^3}$			

Original Function	Rewrite	Differentiate	Simplify
28. $y = \frac{\pi}{(3x)^2}$			
29. $y = \frac{\sqrt{x}}{x}$			
30. $y = \frac{4}{x^{-3}}$			

Finding the Slope of a Graph In Exercises 31–38, find the slope of the graph of the function at the given point. Use the *derivative* feature of a graphing utility to confirm your results.

Function	Point
31. $f(x) = \frac{8}{x^2}$	(2, 2)
32. $f(t) = 2 - \frac{4}{t}$	(4, 1)
33. $f(x) = -\frac{1}{2} + \frac{7}{5}x^3$	(0, $-\frac{1}{2}$)
34. $y = 2x^4 - 3$	(1, -1)
35. $y = (4x + 1)^2$	(0, 1)
36. $f(x) = 2(x - 4)^2$	(2, 8)
37. $f(\theta) = 4 \sin \theta - \theta$	(0, 0)
38. $g(t) = -2 \cos t + 5$	(π , 7)

Finding a Derivative In Exercises 39–52, find the derivative of the function.

39. $f(x) = x^2 + 5 - 3x^{-2}$	40. $f(x) = x^3 - 2x + 3x^{-3}$
41. $g(t) = t^2 - \frac{4}{t^3}$	42. $f(x) = 8x + \frac{3}{x^2}$
43. $f(x) = \frac{4x^3 + 3x^2}{x}$	44. $f(x) = \frac{2x^4 - x}{x^3}$
45. $f(x) = \frac{x^3 - 3x^2 + 4}{x^2}$	46. $h(x) = \frac{4x^3 + 2x + 5}{x}$
47. $y = x(x^2 + 1)$	48. $y = x^2(2x^2 - 3x)$
49. $f(x) = \sqrt{x} - 6\sqrt[3]{x}$	50. $f(t) = t^{2/3} - t^{1/3} + 4$
51. $f(x) = 6\sqrt{x} + 5 \cos x$	52. $f(x) = \frac{2}{\sqrt[3]{x}} + 3 \cos x$



Finding an Equation of a Tangent Line In Exercises 53–56, (a) find an equation of the tangent line to the graph of f at the given point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the *derivative* feature of a graphing utility to confirm your results.

Function	Point
53. $y = x^4 - 3x^2 + 2$	(1, 0)
54. $y = x^3 - 3x$	(2, 2)
55. $f(x) = \frac{2}{\sqrt[4]{x^3}}$	(1, 2)
56. $y = (x - 2)(x^2 + 3x)$	(1, -4)

Horizontal Tangent Line In Exercises 57–62, determine the point(s) (if any) at which the graph of the function has a horizontal tangent line.

57. $y = x^4 - 2x^2 + 3$

58. $y = x^3 + x$

59. $y = \frac{1}{x^2}$

60. $y = x^2 + 9$

61. $y = x + \sin x, \quad 0 \leq x < 2\pi$

62. $y = \sqrt{3}x + 2 \cos x, \quad 0 \leq x < 2\pi$

Finding a Value In Exercises 63–68, find k such that the line is tangent to the graph of the function.

Function	Line
63. $f(x) = k - x^2$	$y = -6x + 1$

64. $f(x) = kx^2$	$y = -2x + 3$
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65. $f(x) = \frac{k}{x}$	$y = -\frac{3}{4}x + 3$
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66. $f(x) = k\sqrt{x}$	$y = x + 4$
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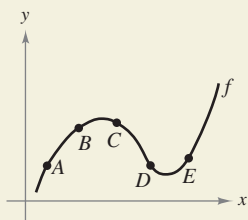
67. $f(x) = kx^3$	$y = x + 1$
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68. $f(x) = kx^4$	$y = 4x - 1$
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69. **Sketching a Graph** Sketch the graph of a function f such that $f' > 0$ for all x and the rate of change of the function is decreasing.



70. HOW DO YOU SEE IT? Use the graph of f to answer each question. To print an enlarged copy of the graph, go to *MathGraphs.com*.



- Between which two consecutive points is the average rate of change of the function greatest?
- Is the average rate of change of the function between A and B greater than or less than the instantaneous rate of change at B?
- Sketch a tangent line to the graph between C and D such that the slope of the tangent line is the same as the average rate of change of the function between C and D.

WRITING ABOUT CONCEPTS

Exploring a Relationship In Exercises 71–74, the relationship between f and g is given. Explain the relationship between f' and g' .

71. $g(x) = f(x) + 6$

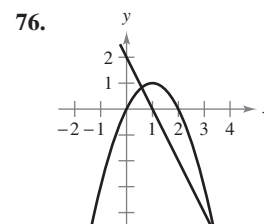
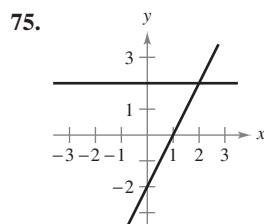
72. $g(x) = 2f(x)$

73. $g(x) = -5f(x)$

74. $g(x) = 3f(x) - 1$

WRITING ABOUT CONCEPTS (continued)

A Function and Its Derivative In Exercises 75 and 76, the graphs of a function f and its derivative f' are shown in the same set of coordinate axes. Label the graphs as f or f' and write a short paragraph stating the criteria you used in making your selection. To print an enlarged copy of the graph, go to *MathGraphs.com*.



77. **Finding Equations of Tangent Lines** Sketch the graphs of $y = x^2$ and $y = -x^2 + 6x - 5$, and sketch the two lines that are tangent to both graphs. Find equations of these lines.

78. **Tangent Lines** Show that the graphs of the two equations

$$y = x \quad \text{and} \quad y = \frac{1}{x}$$

have tangent lines that are perpendicular to each other at their point of intersection.

79. **Tangent Line** Show that the graph of the function

$$f(x) = 3x + \sin x + 2$$

does not have a horizontal tangent line.

80. **Tangent Line** Show that the graph of the function

$$f(x) = x^5 + 3x^3 + 5x$$

does not have a tangent line with a slope of 3.

Finding an Equation of a Tangent Line In Exercises 81 and 82, find an equation of the tangent line to the graph of the function f through the point (x_0, y_0) not on the graph. To find the point of tangency (x, y) on the graph of f , solve the equation

$$f'(x) = \frac{y_0 - y}{x_0 - x}.$$

81. $f(x) = \sqrt{x}$

82. $f(x) = \frac{2}{x}$

$$(x_0, y_0) = (-4, 0)$$

$$(x_0, y_0) = (5, 0)$$



83. **Linear Approximation** Use a graphing utility with a square window setting to zoom in on the graph of

$$f(x) = 4 - \frac{1}{2}x^2$$

to approximate $f'(1)$. Use the derivative to find $f'(1)$.



84. **Linear Approximation** Use a graphing utility with a square window setting to zoom in on the graph of

$$f(x) = 4\sqrt{x} + 1$$

to approximate $f'(4)$. Use the derivative to find $f'(4)$.



85. Linear Approximation Consider the function $f(x) = x^{3/2}$ with the solution point $(4, 8)$.

- (a) Use a graphing utility to graph f . Use the *zoom* feature to obtain successive magnifications of the graph in the neighborhood of the point $(4, 8)$. After zooming in a few times, the graph should appear nearly linear. Use the *trace* feature to determine the coordinates of a point near $(4, 8)$. Find an equation of the secant line $S(x)$ through the two points.

- (b) Find the equation of the line

$$T(x) = f'(4)(x - 4) + f(4)$$

tangent to the graph of f passing through the given point. Why are the linear functions S and T nearly the same?

- (c) Use a graphing utility to graph f and T in the same set of coordinate axes. Note that T is a good approximation of f when x is close to 4. What happens to the accuracy of the approximation as you move farther away from the point of tangency?
- (d) Demonstrate the conclusion in part (c) by completing the table.

Δx	-3	-2	-1	-0.5	-0.1	0
$f(4 + \Delta x)$						
$T(4 + \Delta x)$						

Δx	0.1	0.5	1	2	3
$f(4 + \Delta x)$					
$T(4 + \Delta x)$					



86. Linear Approximation Repeat Exercise 85 for the function $f(x) = x^3$, where $T(x)$ is the line tangent to the graph at the point $(1, 1)$. Explain why the accuracy of the linear approximation decreases more rapidly than in Exercise 85.

True or False? In Exercises 87–92, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

87. If $f'(x) = g'(x)$, then $f(x) = g(x)$.
88. If $f(x) = g(x) + c$, then $f'(x) = g'(x)$.
89. If $y = \pi^2$, then $dy/dx = 2\pi$.
90. If $y = x/\pi$, then $dy/dx = 1/\pi$.
91. If $g(x) = 3f(x)$, then $g'(x) = 3f'(x)$.
92. If $f(x) = \frac{1}{x^n}$, then $f'(x) = \frac{1}{nx^{n-1}}$.

Finding Rates of Change In Exercises 93–96, find the average rate of change of the function over the given interval. Compare this average rate of change with the instantaneous rates of change at the endpoints of the interval.

93. $f(t) = 4t + 5$, $[1, 2]$ 94. $f(t) = t^2 - 7$, $[3, 3.1]$

95. $f(x) = \frac{-1}{x}$, $[1, 2]$ 96. $f(x) = \sin x$, $\left[0, \frac{\pi}{6}\right]$

Vertical Motion In Exercises 97 and 98, use the position function $s(t) = -16t^2 + v_0t + s_0$ for free-falling objects.

97. A silver dollar is dropped from the top of a building that is 1362 feet tall.

- (a) Determine the position and velocity functions for the coin.
- (b) Determine the average velocity on the interval $[1, 2]$.
- (c) Find the instantaneous velocities when $t = 1$ and $t = 2$.
- (d) Find the time required for the coin to reach ground level.
- (e) Find the velocity of the coin at impact.

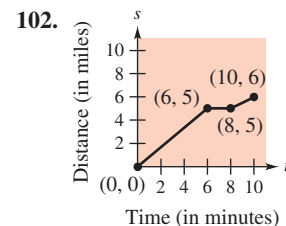
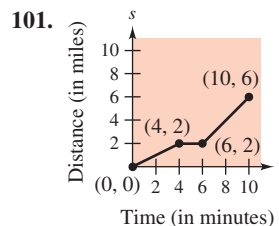
98. A ball is thrown straight down from the top of a 220-foot building with an initial velocity of -22 feet per second. What is its velocity after 3 seconds? What is its velocity after falling 108 feet?

Vertical Motion In Exercises 99 and 100, use the position function $s(t) = -4.9t^2 + v_0t + s_0$ for free-falling objects.

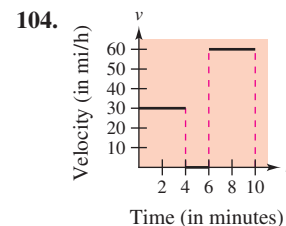
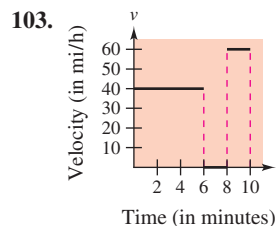
99. A projectile is shot upward from the surface of Earth with an initial velocity of 120 meters per second. What is its velocity after 5 seconds? After 10 seconds?

100. To estimate the height of a building, a stone is dropped from the top of the building into a pool of water at ground level. The splash is seen 5.6 seconds after the stone is dropped. What is the height of the building?

Think About It In Exercises 101 and 102, the graph of a position function is shown. It represents the distance in miles that a person drives during a 10-minute trip to work. Make a sketch of the corresponding velocity function.



Think About It In Exercises 103 and 104, the graph of a velocity function is shown. It represents the velocity in miles per hour during a 10-minute trip to work. Make a sketch of the corresponding position function.

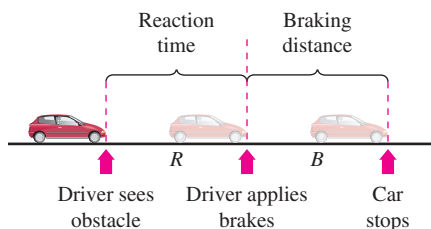


105. **Volume** The volume of a cube with sides of length s is given by $V = s^3$. Find the rate of change of the volume with respect to s when $s = 6$ centimeters.

106. **Area** The area of a square with sides of length s is given by $A = s^2$. Find the rate of change of the area with respect to s when $s = 6$ meters.

107. Modeling Data

The stopping distance of an automobile, on dry, level pavement, traveling at a speed v (in kilometers per hour) is the distance R (in meters) the car travels during the reaction time of the driver plus the distance B (in meters) the car travels after the brakes are applied (see figure). The table shows the results of an experiment.



Speed, v	20	40	60	80	100
Reaction Time Distance, R	8.3	16.7	25.0	33.3	41.7
Braking Time Distance, B	2.3	9.0	20.2	35.8	55.9

- Use the regression capabilities of a graphing utility to find a linear model for reaction time distance R .
- Use the regression capabilities of a graphing utility to find a quadratic model for braking time distance B .
- Determine the polynomial giving the total stopping distance T .
- Use a graphing utility to graph the functions R , B , and T in the same viewing window.
- Find the derivative of T and the rates of change of the total stopping distance for $v = 40$, $v = 80$, and $v = 100$.
- Use the results of this exercise to draw conclusions about the total stopping distance as speed increases.



- 108. Fuel Cost** A car is driven 15,000 miles a year and gets x miles per gallon. Assume that the average fuel cost is \$3.48 per gallon. Find the annual cost of fuel C as a function of x and use this function to complete the table.

x	10	15	20	25	30	35	40
C							
dC/dx							

Who would benefit more from a one-mile-per-gallon increase in fuel efficiency—the driver of a car that gets 15 miles per gallon, or the driver of a car that gets 35 miles per gallon? Explain.

- 109. Velocity** Verify that the average velocity over the time interval $[t_0 - \Delta t, t_0 + \Delta t]$ is the same as the instantaneous velocity at $t = t_0$ for the position function

$$s(t) = -\frac{1}{2}at^2 + c.$$

- 110. Inventory Management** The annual inventory cost C for a manufacturer is

$$C = \frac{1,008,000}{Q} + 6.3Q$$

where Q is the order size when the inventory is replenished. Find the change in annual cost when Q is increased from 350 to 351, and compare this with the instantaneous rate of change when $Q = 350$.

- 111. Finding an Equation of a Parabola** Find an equation of the parabola $y = ax^2 + bx + c$ that passes through $(0, 1)$ and is tangent to the line $y = x - 1$ at $(1, 0)$.

- 112. Proof** Let (a, b) be an arbitrary point on the graph of $y = 1/x$, $x > 0$. Prove that the area of the triangle formed by the tangent line through (a, b) and the coordinate axes is 2.

- 113. Finding Equation(s) of Tangent Line(s)** Find the equation(s) of the tangent line(s) to the graph of the curve $y = x^3 - 9x$ through the point $(1, -9)$ not on the graph.

- 114. Finding Equation(s) of Tangent Line(s)** Find the equation(s) of the tangent line(s) to the graph of the parabola $y = x^2$ through the given point not on the graph.

- (a) $(0, a)$ (b) $(a, 0)$

Are there any restrictions on the constant a ?

Making a Function Differentiable In Exercises 115 and 116, find a and b such that f is differentiable everywhere.

115. $f(x) = \begin{cases} ax^3, & x \leq 2 \\ x^2 + b, & x > 2 \end{cases}$

116. $f(x) = \begin{cases} \cos x, & x < 0 \\ ax + b, & x \geq 0 \end{cases}$

- 117. Determining Differentiability** Where are the functions $f_1(x) = |\sin x|$ and $f_2(x) = \sin |x|$ differentiable?

- 118. Proof** Prove that $\frac{d}{dx}[\cos x] = -\sin x$.

FOR FURTHER INFORMATION For a geometric interpretation of the derivatives of trigonometric functions, see the article “Sines and Cosines of the Times” by Victor J. Katz in *Math Horizons*. To view this article, go to MathArticles.com.

PUTNAM EXAM CHALLENGE

- 119.** Find all differentiable functions $f: \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f'(x) = \frac{f(x+n) - f(x)}{n}$$

for all real numbers x and all positive integers n .

This problem was composed by the Committee on the Putnam Prize Competition. © The Mathematical Association of America. All rights reserved.